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THE SCATTERING OF X-RAYS AND BRAGG'S LAW

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In 1914 Darwin[1] showed that when the classical electromagnetic theory of light is applied to the regular reflection of X-rays by a crystal a small departure from Bragg's law is demanded if there is a refractive index for X-rays in the crystal. The theory further demands that the refractive index shall be $1-\mu$, where μ is a small positive quantity. Stenström[2] was the first to show a departure from Bragg's law experimentally. He found that the wave-length of a spectrum line as determined by applying Bragg's law to the second order reflection was slightly less than that as determined by the first order reflection, the same crystal being used in each case. This is as it should be according to Darwin's theory. Stenström's result has been confirmed by Siegbahn,[3] Hjalmar,[4] and by Duane and Patterson.[5] Confirmation by another method of an index of refraction for X-rays in solids of a value less than unity has also been obtained by A. H. Compton[6] in his experiments on total reflection.

A few months ago A. H. Compton published a quantum theory of the scattering of X-rays[7] in which he showed that when X-rays are scattered by free electrons a change of wave-length in the scattered rays should be expected. This change of wave-length has been observed experimentally by Compton[8] for Mo Kα rays scattered by carbon. It occurred to Mr. C. H. Eckart and the present writer that, if crystalline reflection of X-rays is a special case of scattering, then the reflected ray should be of longer wave-length than the incident ray. Making this assumption the effect on Bragg's law was discussed in a letter to *Nature*.[9] It was there shown that the experimental departures from Bragg's law could be explained by this assumption of a change of wave-length on reflection. The same explanation was made later by Wolfers[10] and still later by Hulburt.[11] Hulburt, however, remarks that the theoretical deviation from Bragg's law is of the same order of magnitude as the experimental deviation but is of the wrong sign, while Jauncey and Eckart[9] found the theory to give

the same sign as the experiments. There is a reason for this disagreement between Hulbert and Jauncey and Eckart, and we shall now proceed to consider the theory in detail.

Let the incident and reflected wave-lengths be λ_1 and λ_2, respectively, the grazing angles of incidence and reflection being θ_1 and θ_2, respectively. Then for reflection from a single plane of atoms we have

$$\cos \theta_1/\lambda_1 = \cos \theta_2/\lambda_2 \qquad (1)$$

while for reflection from two consecutive planes of atoms

$$d \sin \theta_1/\lambda_1 + d \sin \theta_2/\lambda_2 = n \qquad (2)$$

where $2n\pi$ is the difference of phase between the rays reflected from the first and second planes, respectively. Further, the total angle of deviation between the incident and reflected rays is $(\theta_1 + \theta_2)$ and applying Compton's change of wave-length formula[7] we have

$$\lambda_2 = \lambda_1 + 2\gamma \sin^2 (\theta_1 + \theta_2)/2 \qquad (3)$$

where $\gamma = h/m\,c = 0.024\ A$. These three equations may be solved for λ_1 in terms of d, n and θ_1 or in terms of d, n and $(\theta_1 + \theta_2)/2$. To facilitate the first solution let us put $x = \lambda_2/\lambda_1$ and from eqs. (1) and (3) we obtain an equation for x in terms of θ_1 and α where $\alpha = \gamma/\lambda_1$. This equation is a quadratic with the solutions $x = 1$ and

$$x = (1 + 2\alpha + \alpha^2 \cos^2 \theta_1)/(1 + 2\alpha \cos^2 \theta_1 + \alpha^2 \cos^2 \theta_1). \qquad (4)$$

The meaning of the solution $x = 1$ is that there is a Laue spot in the forward direction of the primary beam. However, instead of expressing λ_1 in terms of d, n, and θ_1, we can obtain a relation between λ_1 and L', where L' is the apparent wave-length given by Bragg's law $nL' = 2d \sin \theta_1$, when the grazing angle of incidence θ_1 is measured. This relation is

$$L' = \lambda_1 + \gamma - \gamma^2 \sin^2 \theta_1/(\lambda_1 + \gamma). \qquad (5)$$

From eq. (5) we see that L' becomes less for greater values of θ_1, that is, for higher orders of reflection and thus gives a deviation from Bragg's law of the right sign. The deviation is also of the right order of magnitude as shown in table I. The experimental values are taken from the paper of Duane and Patterson,[5] who measured θ_1 using a crystal of calcite.

TABLE I

L' A	M	N	L'M − L'N EXPERIMENT A	L'M − L'N THEORY A
1.473	1	2	0.00015	0.00007
1.279	1	3	0.00025	0.00017
1.096	1	2	0.00006	0.00005

In the above table m and n are the orders of reflection while $L'm$ is the apparent wave-length when calculated by Bragg's law for the m th order. Now, however, let us examine the relation between L'' and λ_1, where L'' is

the apparent wave-length given by Bragg's law when $(\theta_1 + \theta_3)$ is measured. Let $(\theta_1 + \theta_2)/2 = \beta$ so that $nL'' = 2d \sin\beta$ and we find

$$L'' = \lambda_1(1 + 2\alpha \sin^2\beta)/\sqrt{(1 + 2\alpha \sin^2\beta + \alpha^2 \sin^2\beta)}, \quad (6)$$

or

$$L'' = \lambda_1 + \gamma \sin^2\beta \quad (7)$$

to the first power of γ. From this it appears that L'' becomes larger for higher orders of reflection, which is contrary to experiment. However, if the experimental values of Hjalmar,[4] who used a crystal of gypsum and measured $(\theta_1 + \theta_2)$, are used we obtain table II.

TABLE II

L'' A	M	N	$L''M - L''N$ EXPERIMENT A	$L''M - L''N$ THEORY A
1.389	1	2	0.00045	—0.0006
2.279	1	2	0.0014	—0.0016
2.279	1	3	0.0016	—0.0043

The theoretical values are of the right order of magnitude. This is the result obtained by Wolfers,[10] but he states that his theoretical values are the right sign. My values, however, are of the wrong sign, which result has also been reported by Hulburt.[11] Furthermore θ_1 and β are approximately equal so we may apply eq. (5) to the experiments of Hjalmar. We find, however, that although eq. (5) gives values of the right sign yet they differ from the experimental values of table II in order of magnitude, being some 0.02 times these values. It would seem therefore that the agreement between the experimental and the theoretical values of tables I and II is purely fortuitous.

Still further let us compare eqs. (5) and (7) and we find

$$L' - L'' = \gamma \cos^2\theta_1 \quad (8)$$

to the first power of γ. Or $L' - L'' = 0.024\ A$ approximately for the lower orders of reflection. Duane and Patterson[5] have compared their results with those of Overn,[12] who measured $(\theta_1 + \theta_2)$ for the same tungsten lines. Differences varying from $-0.0023\ A$ to $+0.0011\ A$ were found. These are much less than the $0.024\ A$ predicted by the present theory. It is seen therefore that the experimental deviations from Bragg's law cannot be explained on the assumption of a change of wave-length on reflection. We are therefore forced to accept Darwin's theory of a refractive index as providing the only available explanation of the deviations from Bragg's law.

[1] C. G. Darwin, *Phil. Mag.*, **27**, 318 (1914).
[2] Stenström, *Doctor's Dissertation*, Lund, 1919.
[3] M. Siegbahn, *C. R., Paris Acad. Sci.*, **173**, 1350.
[4] E. Hjalmar, *Zeit. Physik.*, **1**, 439 (1920).
[5] Duane and Patterson, *Physic. Rev.*, **16**, 526 (1920).
[6] A. H. Compton, *Phil. Mag.*, **45**, 1121 (1923).

[7] A. H. Compton, *Physic. Rev.*, **21**, 483 (1923).
[8] A. H. Compton, *Ibid.*, **22**, 409 (1923).
[9] Jauncey and Eckart, *Nature*, **112**, 325 (1923).
[10] M. F. Wolfers, *C. R., Paris Acad. Sci.*, **177**, 759 (1923).
[11] E. O. Hulburt, *Phys. Soc. Meeting*, Chicago, 1923.
[12] O. B. Overn, *Physic. Rev.*, **14**, 137 (1919).

REFRACTION OF X-RAYS IN PYRITES

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In all careful determinations of wave-lengths of X-rays by crystal reflection it has been found that the relation

$$n\lambda = 2d \sin \Theta$$

does not hold accurately for the several orders. The departure from this law has been rightly ascribed by Stenstrom[1] to refraction in the crystal. This refraction has been observed by Hjalmar,[2] Davis and Terrill[3] and others, and has been directly confirmed by experiments of Compton on total reflection. If the angles are measured with respect to the crystal surface, the index of refraction μ is expressed by

$$\mu = \frac{\cos \Theta_1}{\cos (\Theta_0 - \varphi)} \qquad (1)$$

where Θ_1 is the angle of incident rays to the surface outside the crystal and φ is the angle of the surface to the molecular planes and Θ_0 is the angle of the X-ray beam to the molecular planes inside the crystal.

Placing $\mu = 1 - \delta$, Stenstrom has derived the following expression for δ, for the case when $\varphi = 0$.

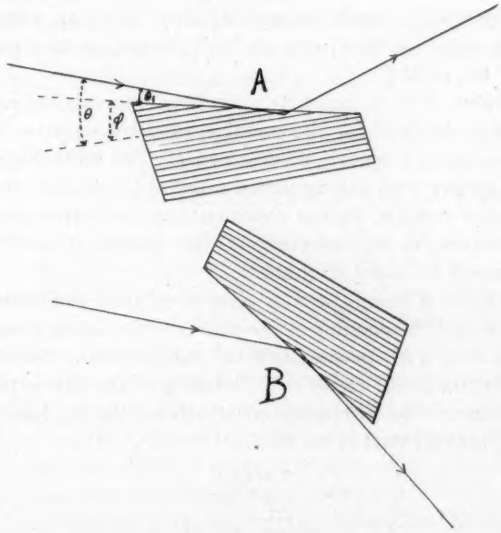
$$\delta = \frac{\left(\frac{\sin \Theta_m}{m}\right)^2 - \left(\frac{\sin \Theta_n}{n}\right)^2}{2\left(\frac{\cos \Theta_m}{m}\right)^2 - 2\left(\frac{\cos \Theta_n}{n}\right)^2}, \qquad (2)$$

where Θ_m, Θ_n are the observed glancing angles at the orders m and n.

The bending of the rays by refraction as they pass through the surface in the case of the natural cleavage surface is small. The bending for Mo, $K_{\alpha 1}$ radiation in calcite for instance is found to be about 3″ arc. Since it is difficult to measure such a small effect accurately, the following method was proposed to increase the bending due to refraction. As in the case

of light, the bending is increased as the angle of the ray to the surface becomes very small. The crystal was consequently cut and optically polished so that the surface made an angle φ to the planes as shown in the figure.

The source of the radiation was a water cooled molybdenum tube kindly supplied by the Research Laboratory of the General Electric Company. The measurements were made with an especially designed ionization spectrometer. This spectrometer is provided with an accurate tangent worm and hand wheel by which small angles can be turned through and read with great accuracy. The worm and hand wheel were calibrated by means of an optical lever device, the distance of the scale from the mirror mounted on the crystal table being about 55 feet.



For the purpose of the accurate measurement of wave-length, it is necessary to turn the crystal through an angle $\beta = (180° - 2$ the glancing angle). This cannot be done with sufficient accuracy by vernier readings. It was accomplished by placing an optically plane-parallel interferometer mirror on the crystal table just above the crystal. A large telescope and brightly illuminated scale were placed at a distance of about 55 feet. Since the mirror was plane-parallel the crystal could be rotated through an angle of 180° with an accuracy of one second. The additional angle could be measured by the optically calibrated tangent worm and hand wheel. The experiments were first performed with the natural (100) face of a very perfect crystal of pyrites. Measurements were made at the 1st

and 4th orders. The value of δ was directly calculated by equation 2. The value so obtained is given in the last column of table 1 for $\varphi = 0$.

The crystal was then ground and polished so that the surface made an angle $\varphi = 6°31'57.5''$ with the reflecting planes. The position of the crystal and path of rays is shown in the figure (A). The crystal is then turned through an angle $\beta = (180-2\alpha)$, at which reflection is then observed. (Position B.) Measurements could be successfully made only at the 1st order for position B, the 4th order being very weak on account of absorption. It may be shown that δ may be expressed in terms of the 1st order readings by the equation

$$\delta = \frac{(\sin\alpha - \sin\Theta_0)(\sin^2\alpha - \sin^2\varphi),}{\sin\alpha\cos^2\alpha}, \qquad (3)$$

where $\alpha = \frac{1}{2}(180° - \text{angle turned through})$ in going from position A to position B, and Θ_0 is the true angle that the rays make with the crystal planes inside the crystal.

The true value of Θ_0 is found by successive approximations as follows: The value of δ obtained for the uncut crystal by equation 2 is used in equation 1 to obtain a tentative value of Θ_0. The tentative Θ_0 is used in equation 3 together with the measured α and φ to calculate a tentative δ. This tentative δ is again used in equation 1 to calculate a more accurate Θ_0. These successive approximations are continued until sufficiently accurate values of Θ_0 and δ are obtained.

The final values of δ and the true value of Θ_0 are given in the table. It will be noticed that the bending of the rays $\Theta_1 - (\Theta_0 - \varphi)$ on passing through the surface is only 3.6″ for a natural crystal face while it is 39″ for $\varphi = 6° 31'57.5''$ and for $\varphi = 7° 18'39''$, the bending of the ray is much greater being 159″ of arc. The corresponding results for the K_{β_1} line ($\lambda = 0.6311$) are given in the lower part of the table.

TABLE I

$\lambda = 0.7078$

φ	$\alpha = \frac{1}{2}(180° - \beta)$	$\Theta_1 - (\Theta_0 - \varphi)$	$\delta(\times 10^6)$
0°	7° 31′ 28″	3.6″	4.6
6° 31′ 57.5″	7° 31′ 43.5″	39″	3.26
7° 18′ 39″	7° 32′ 48.5″	159″	3.37

$\Theta_0 = 7° 31' 22.9''$

$\lambda = .6311$

0°	6° 42′ 18.5″	3″	3.87
6° 31′ 57.5″	6° 43′ 41″	160″	2.82

$\Theta_0 = 6° 42' 13.4''$

where φ = angle between crystal surface and planes.
β = angle turned through from position A to B.

Θ_1 = angle of ray to surface outside crystal.
Θ_0 = true angle of ray to planes inside crystal.

It is of interest to compare these results with the dispersion formula derived for ordinary light by H. A. Lorentz.[4] This formula expressed in the usual electromagnetic units may be written

$$\delta = \frac{e^2}{2\pi m}\left(\frac{n_1}{\nu^2-\nu_1^2}+\frac{n_2}{\nu^2-\nu_2^2}+\text{etc.}\right)$$

where ν is the frequency of the incident radiation and n_1, n_2 are the number of electrons per unit volume that have natural frequencies ν_1, ν_2, etc. The calculated value of δ given in table II was obtained by the above formula.

The weighted means of the experimental results are given in table II. The weighting was controlled by the magnitude of the bending $\Theta_1-(\Theta_0-\varphi)$ and the accuracy of the measurement of φ.

TABLE II

λ	(EXP.)	(CALC.)
0.6311	2.82×10^{-6}	2.62×10^{-6}
0.7078	3.33×10^{-6}	3.29×10^{-6}

On account of the greater energy in the K_α line, the measurements for it are probably much more accurate than those for the K_β.

The agreement of the experimental results with the theory of Lorentz is important and suggestive. The theory is based primarily on the principle of resonance. Each term becomes very large as the frequency of the incident radiation approaches the natural frequency of any group of electrons. The critical frequencies of the elements investigated up to the present time are too far removed from the incident frequency to determine whether this resonance effect is present or not. It is hoped in the near future to determine the refractive effect of iron pyrites on the K_α radiation of copper. The first term of the Lorentz dispersion formula becomes quite large for this case.

[1] W. Stenstrom. "Untersuchungen der Rontgenspektra." *Dissertation*, Lund, 1919.
[2] Hjalmar, *Zeit. Physik.*, **7**, 1921.
[3] *Proc. Nat. Acad. Sci.*, Dec. 1922.
[4] *Theory of Electrons*, page 159.

THE EFFECTS OF REPLACING THE CEPHALIC END OF THE EMBRYONIC SPINAL CORD BY AN EXTRANEOUS MEDULLA IN AMBLYSTOMA

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Certain aspects of the nature of stimuli affecting sensory and motor nerve proliferation in developing Amphibian larvae have been considered in several previous communications.[1] The experiments involved consisted chiefly in the transplantation of limbs and units of spinal cord in Amblystoma embryos. The results obtained have indicated, in brief, that the extent of proliferation in somatic sensory centers (spinal ganglia) is markedly effected by experimentally decreasing or increasing the peripheral integumentary areas.[2] On the other hand, somatic motor nerve centers in the spinal cord failed to exhibit any appreciable developmental responses to decreases or to increases in the peripheral somatic musculature, even though in functional connection with the latter.

The deduction has been made that the amount of cellular proliferation throughout the brachial level of the spinal cord (third, fourth and fifth segments) is not primarily determined by the functional activity of the shoulder and limb musculature. It appears rather to be dominated by stimuli reaching these levels through caudally directed correlation fibers which normally terminate at this region of the cord for the control of the limbs. This appears to have been borne out in certain spinal cord transplantation experiments. These have shown that the seventh, eighth and ninth segments of the cord, when in connection with a transplanted limb, do not exhibit any increased cellular proliferation in response to the added need at the periphery. However, when these same segments are transplanted into the brachial level (third, fourth and fifth segments), they undergo a marked cellular hyperplasia so that the extent of development attained in these transplanted segments is approximately equal to that which characterizes the normal brachial region.[3] Recent experiments, not yet published, indicate that the same holds true in those cases where, in addition, the limb has been removed.

In the normal brachial level of the spinal cord there exists a gradient of cellular proliferation as indicated in figure 3 (SS3, SS4, SS5). The cephalic end (third segment) of this brachial region undergoes greater proliferation than the caudal end (fifth segment). However, when the brachial level is reversed end for end the fifth segment in its new situation, undergoes a degree of differentiation which is approximately the same as that in the normal third. Conversely, the third segment in its new position under-

goes a decreased proliferation equivalent to that normally attained in the intact fifth segment.[4]

The cumulative evidence thus offered has indicated that cellular proliferation in the cord is under the influence of central pathways descending from the brain for discharge into the somatic motor centers of the spinal cord. In addition to the medial longitudinal fasciculus, Herrick[5] has demonstrated the presence of a tract termed by him the bulbo-spinal tract, the cell bodies of which lie in the general motor tegmentum of the medulla and the axones of which are directed caudally into the ventral funiculi of the same and the opposite side of the cord.

Attempts were made several years ago to disconnect functionally these descending fibers from the spinal cord by interposing mechanical obstacles between the brain and the spinal cord. These were unsuccessful. However, during the past spring a new type of experiment was successfully carried out in the attempt to show more directly whether or not the descending tracts already considered influence cellular proliferation in the spinal cord in a way ascribed to them.

These experiments were carried out as follows: The first five segments

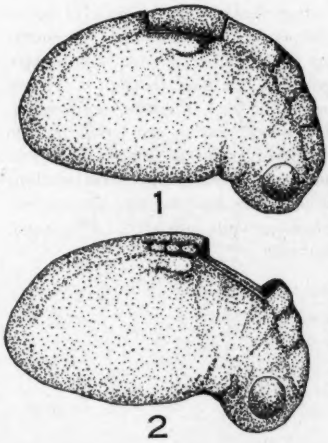

FIGURE 1

Outline drawing of Amblystoma embryo in stage 25, showing replacement of the first five segments of spinal cord by a unit of neural tube comprising the major portion of the medulla and the first two spinal segments from another embryo. X 10 (cf. figure 2).

FIGURE 2

Outline drawing of Amblystoma embryo in stage 25, showing region of neural tube (major portion of medulla and first two spinal segments) which was excised and transplanted to embryo shown in figure 1. X 10.

of the spinal cord were excised, of which the third, fourth and fifth segments constitute the brachial level. To this region was transplanted as a unit the medulla (exclusive of the cephalic portion) and the first two spinal segments of the cord from another individual (v. figures 1 and 2). The transplanted medulla occupied approximately that portion of the spinal cord normally composing the first three segments, whereas the first and second segments occupied the regions normally made up of the fourth and fifth segments, respectively.

Of one hundred fifty experimental animals only eight survived, of which four were considerably deformed. A heavy infection of Saprolegnia, however, in a group of fifty experimental animals undoubtedly was responsible for a considerable number of deaths. Four animals were reared for forty-five to fifty day and their reactions were closely observed.[6] At this period they were photographed and preserved. Corrosive sublimate

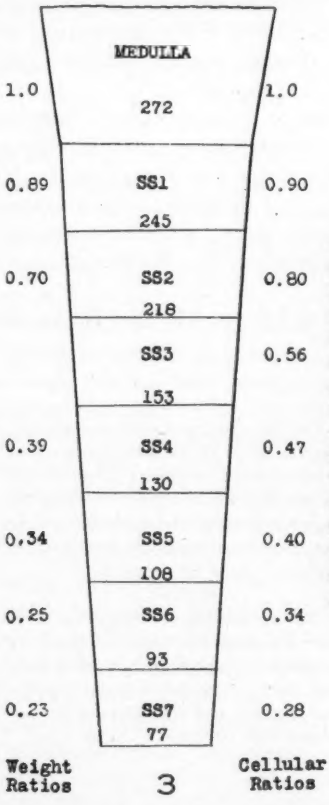

FIGURE 3

A diagrammatic plan of the cellular region of the central nervous system of a normal Amblystoma larva from the level of the acoustic (eighth cranial) nerve to the caudal limit of the seventh spinal segment. The central figures express the average number of nerve cells per section as obtained from counting the number in ten consecutive unilateral sections through a known region for each segment represented. The weight ratios are based on the weight in grams of unassembled wax models of thirty consecutive sections through the medulla and the various spinal segments indicated. SS—spinal segments.

acetic and vom Rath fixatives were used and the animals were sectioned serially at 10μ.

The method of study consisted in determining the extent of development characterizing the various levels of the central nervous system from the medulla to that of the seventh nerve inclusive, in both normal and experimental animals. Unassembled wax models comprising thirty consecutive transverse sections from each segment of the cord were weighed. The weights were compared with the weight of thirty sections taken from

a definite level in the medulla (level of the acoustic nerve) and the ratios obtained. Similarly, counts were made of the cells in ten consecutive transverse sections throughout the same levels in the normal and the experimental animals and the ratios likewise obtained. The results of these estimates are given in figures 3 and 4. The outstanding feature consists in the remarkable increased cellular proliferation in the trans-

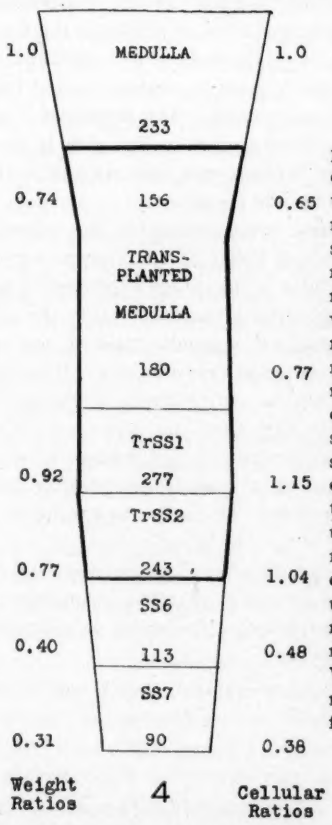

FIGURE 4

A diagrammatic plan of the cellular portion of the central nervous system of Amblystoma larva TrBrSc 1 from the level of the acoustic (eighth cranial) nerve to the caudal level of the seventh spinal segment. The first five segments of the cord were excised and replaced by a unit of neural tube involving the greater portion of the medulla and the first two spinal segments (TrSS1 and TrSS2). The added medulla in this case occupies the levels of the normal first, second and third segments. The transplanted first and second segments occupy the regions of the normal fourth and fifth, respectively (cf. figure 3). The central figures express the average number of nerve cells per section as obtained from counting the number in ten consecutive unilateral sections through the same regions as employed in the study of the normal individual (cf. figure 3). Weight ratios obtained in the same manner as outlined in explanation of figure 3.

planted first and second segments (TrSS1 and TrSS2). Under normal conditions the extent of proliferation in these segments is less than that found in the given region selected from the medulla (figure 3). In figure 4 it is also seen that cellular proliferation in the transplanted first and second segments not only exceeds that found in the transplanted medulla (with which they were transplanted), but the number of cells in these segments is greater than that found in the corresponding region of the

normal medulla. Further, it is to be noted that the transplanted first and second segments with their greatly increased cellular state are on a level of the cord corresponding to the fourth and fifth segments where, under normal conditions the degree of cellular development falls greatly below that normally found in the intact first and second segments (figure 3).

Even though the transplanted second segment in the experimental animal marks the caudal limit of the brachial level, it is seen in the figures that the average number of cells per section throughout the sixth and seventh segments of the experimental case is greater in relation to that found in the normal medulla than in the normal animal. The difference is seen to be greater in the sixth than in the seventh segment, which is to be expected since under normal conditions of development, cellular proliferation throughout the sixth segment is higher than in the seventh.

A comparison of the results obtained from estimating the extent of cellular development throughout identical levels of the central nervous system from the caudal end of the medulla to the seventh segment in both the normal and the experimental larvae, strongly suggest that in the latter the hyperplastic development in the spinal segments studied has been brought about under the influence of an additional medulla. The exact nature of this supposed influence is not as yet understood, though the evidence at hand strongly favors the idea that the increased cellular proliferation in the segments of the cord caudal to the transplanted medulla has been brought about under the stimulation made possible by an increased number of caudally growing projection pathways arising in the extra medulla.

A more complete account of these experiments will be forthcoming with a discussion of the results in reference to Bok's[7] so-called "Stimulogenous Fibrillation Concept," which has been referred to briefly in connection with former experiments on the spinal cord.[8]

[1] Detwiler, S. R., (*a*) These PROCEEDINGS, **6** (96–101); (*b*) *Ibid.*, **6** (695–700); (*c*) *J. Exp. Zoöl.*, **37** (339–393); (*d*) *Ibid.*, **38** (293–321).

[2] See 1 (*a*).

[3] See 1 (*c*), table 2 and figures 9 and 10.

[4] See 1 (*d*), table 3.

[5] Herrick, C. J. 1914. "The Medulla Oblongata of Larval Ambylstoma." *J. Comp. Neur.*, v. **24** (343–427).

[6] An account of the reactions will be considered in a more complete account of the work.

[7] Bok, S. T. 1915. "Die Entwicklung der Hirnnerven und iher zentralen Bahnen. Die Stimulogene Fibrillation." *Folia Neurobiolog.*, **9** (475–565).

[8] See 1 (*d*).

SOME UNEXPECTED RESULTS OF THE HETEROPLASTIC TRANSPLANTATION OF LIMBS

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Read before the Academy, Nov. 12, 1923

The embryos of distinct but closely related species of amphibians often differ from one another in certain qualities, such as pigmentation, size, time of appearance of organs, etc., which render them useful in the analysis of the factors of growth and development by means of grafting experiments. Two of our native salamanders, Ambylstoma punctatum Linn. and A. tigrinum Green, afford advantages of this kind in connection with the development of the limbs.

In the spotted salamander, A. punctatum, the fore legs begin to show externally at a comparatively early stage and but little later than the external gills. By the time the yolk is resorbed, they are fully developed both in form and in function. On the other hand, in the tiger salamander (A. tigrinum) the fore limbs at this period are small nodules still, scarcely visible superficially. It is not until some weeks later that they reach the degree of development attained by the fore limbs of the punctatum larva at the close of the embryonic period. With respect to the development of most of the other organs the embryos of these two species show much greater concurrence, but there are several characteristic differences that require brief notice here.

The tigrinum egg is smaller and contains much less yolk, so that this store of food is exhausted sooner than in the punctatum egg. The tigrinum larva consequently begins to feed when quite small, although, except for the difference in the fore limbs, the embryos of the two species are approximately at the same stage of development at this period. During the first part of larval life the rate of growth is not very different in the two species, but after five or six weeks, when they have reached a length of about 4 cm., the tigrinum larvae begin to forge ahead rapidly and ultimately attain a much larger size than the punctatum. Sexually mature larvae (axolotls) that have come under my notice measure from 25 to 32.5 cm. in length. The two longest adult specimens mentioned by Cope[1] measured 25 and 27.5 cm., respectively. In the strain used in the present investigation metamorphosis has not been observed to take place in specimens less than 13 cm. in length, and it often, apparently, does not occur at all, while in A. punctatum it takes place when the larvae are about 5 cm. long. This size may be reached under favorable circumstances in the latter species in about ten or twelve weeks, at which time tigrinum larvae of the same age, kept under similar conditions, are about 6 cm. long,

without, however, showing any sign of metamorphosis. Adult punctatum individuals measure about 16 cm. in length.

For the purpose of investigating the factors concerned in the development and growth of the fore limbs, limb buds were interchanged between embryos of the two species. The grafting was done in the usual way at an early period of development (stages 29–31),[2] before the limb buds could be distinguished as such, and before peripheral nerves or blood vessels were present.

The initial effect of the operation was as expected, especially in the one series of experiments. The punctatum limb implanted on a tigrinum embryo grows rapidly, and is soon far beyond the normal tigrinum limb in development. It does not, however, keep pace with the normal undisturbed punctatum limb of the donor, and it does not during this period acquire its normal powers of movement.

In the reciprocal experiment the tigrinum limb on the punctatum embryo remains very small for a short time, while the normal limb on the opposite side grows out rapidly. In this case, however, the effect of the transplantation soon becomes evident in a positive manner, for the grafted limb begins to grow far more rapidy than it would have, had it been left in place. When the punctatum larva begins to feed (stage 46) the normal limb has three digits with the beginning of the fourth, and it functions perfectly. While the grafted tigrinum limb is less advanced than this, it has, nevertheless, lengthened far beyond the normal tigrinum limb, which is still a nodule, and it shows the notch marking the first two digits, a condition reached by the normal punctatum limb in stage 42. Such grafted limbs even show slight movements at the shoulder, although the normal punctatum limb does not reach this functional condition until stage 44, when the hand is much better developed.

The subsequent behavior of the grafted limbs in both series of experiments is notable for the increasing influence of the host on their growth.

The transplanted punctatum leg lags more and more behind the normal fore leg of the donor. For a time it appears atrophic, and its only movement is a slight twitching at the shoulder. This lack of motility is in itself interesting, for sections of individuals preserved during this period show that the nerves of the limb are well-developed. The absence of normal function is, therefore, probably due to conditions within the central nervous system, corresponding to the rudimentary state of the normal tigrinum limb. In several cases the transplanted leg seemed for a time to be undergoing degeneration. At the same time the tigrinum host grew rapidly, and its own fore leg, in the course of about six or seven weeks from the date of operation, overtook and soon exceeded in size the grafted one (figs. 1 and 2), which remained very small, though it later acquired perfect form and function, all traces of degeneration disappearing. Three such

specimens have been kept through this stage. One (fig. 1) was preserved and sectioned, showing a diminutive shoulder girdle as well as limb. Two are still alive. One of the latter metamorphosed when 13 cm. long, at which time the grafted limb was resorbed to a short stump. The other, now 14.7 cm. long is still a larva. The transplanted limb is very small as compared with the normal limb of the host, but it is now larger than the limb of the punctatum donor, which latter is a much smaller animal, 6.6 cm. long. The function of the grafted limb is perfect. Its shape is that of the donor species (punctatum), but the skin markings and coloration are those of A. tigrinum, showing that extensive pigment migration must have occurred.

Fig. 1.—Larva of Amblystoma tigrinum with a punctatum fore limb (*gr*) on the right side; *n*, normal limb. Exp. NE.10; specimen preserved 76 days after operation.

Fig. 2.—Larva of A. punctatum, the donor of the limb shown in fig. 1. Regeneration of the lost limb has not occurred. Specimen preserved 76 days after operation.

Fig. 3.—Larva of A. punctatum showing a gigantic, but otherwise normal, tigrinum limb (*gr*), grown to this size after having been grafted in the limb bud stage; *n*, normal limb. Exp. NE.13, specimen preserved 72 days after operation.

Fig. 4.—Normal control larva of A. tigrinum of the same age.

All figures magnified 1.4 diameters.

The tigrinum limb grafted to the punctatum larva grows very rapidly after it once begins, so that shortly after the yolk is resorbed it overtakes the normal limb in degree of development, at the same time having become

more massive. It is at first white and almost without pigment cells, like the developing tigrinum leg, whereas the punctatum leg is well-pigmented. The grafted limb soon outstrips the normal one on the opposite side, growing rapidly until the time of metamorphosis, when in the most extreme case its linear dimensions are fully double those of the normal (figs. 3 and 4). These animals are very grotesque in appearance, especially when viewed from the front. The gigantic limbs, however, like the diminutive ones in the reciprocal experiment, are well proportioned and function perfectly. The shoulder girdle and pectoral muscles are likewise enormously developed for the size of the animal. None of the five specimens of this series survived metamorphosis.

Sections show that the constituent parts of the limbs in both kinds of transplantations are proportionately developed. Skeleton, muscles, and nerves are small in the miniature transplanted punctatum limb and large in the giant tigrinum limb on the punctatum larva. However, neither the central nervous system nor the spinal ganglia show hypoplasia in the one case or hyperplasia in the other. This may seem odd in connection with Detwiler's[3] discovery that dimunition or increase in the peripheral area innervated by a given nerve results, respectively, in the hypoplasia or hyperplasia of the ganglion of that nerve. It should be recalled, however, that the limb does not immediately show its ultimate growth powers after either type of transplantation. On the contrary, the early tendencies are opposite to the later ones in this respect, both kinds of limb buds at first manifesting their own inborn characters, so that the punctatum limb on the tigrinum embryo is for some time actually larger than the normal tigrinum limb, and the grafted tigrinum limb requires about a month to overtake the normal punctatum limb in size. In this early formative period, therefore, the tendency would be toward hyperplasia of the spinal ganglia innervating the limb that ultimately remains smaller, and toward hypoplasia of those innervating the limb that ultimately becomes much larger. Apparently the limb does not reach its ultimate disproportionate relation sufficiently early to produce an effect on the sensory nerve centers.

In order to account for the above results it is necessary to assume that two distinct factors, aside from the ordinary nutritional ones, are concerned in the growth of the limbs. One of these, immanent in the cells of the transplanted limb and derived from the general constitutional qualities of the species, may be termed the growth potential (G). This is much greater in A. tigrinum (t), as shown by the much larger ultimate size of individuals of this species in comparison with A. punctatum (p). The other factor may be designated as a regulator (R), since it acts by controlling the rate of growth. It exists not in the individual cells of the transplanted tissue, but in the circulating medium, and is probably a hormone

derived either from the hypophysis or the thyroid gland. Since A. punctatum metamorphoses very early as compared with A. tigrinum, there is justification for the assumption that this regulating factor is more effective in the former species. The relative effect of the four combinations (two normal and two experimental) of these factors, acting until the time of metamorphosis of the punctatum larvae (ten or twelve weeks after grafting), may be expressed by the inequality

$$G_tR_p > G_tR_t > G_pR_p > G_p\mathrm{R}_t$$

What the ultimate relation of these effects would be if the salamanders were kept until they all reached full size remains unknown, because no individuals of the first combination have survived metamorphosis. In other words, we do not know whether the grafted tigrinum leg would ultimately exceed its normal size, or whether it would simply reach this condition in less time and then remain stationary. We do know, however, that the order of the last two terms will be reversed later, for the actual size of the grafted punctatum limb in the oldest specimen under observation is larger than that of the normal control limb of that species. Here, however, there is a likelihood that nutritional factors are concerned, for the tigrinum individual with the grafted leg is very much larger than the punctatum donor, and presumably a larger supply of nutrient material reaches the grafted leg than the leg of the latter.

The foregoing experiments show with great clearness that various influences are involved in growth. However, it is desirable here to emphasize the importance of the constitutional factors, because they are more likely to be overlooked. The grafted limb is subject to exactly the same nutritional and hormonic influences as the normal limb of the same individual, and yet, by virtue of its specific constitution, its growth is enormously greater in the one and correspondingly less in the reciprocal combination than the growth of the normal control limb on the opposite side of the body. Again when a comparison is made between the grafted limb and the control left on the body of the donor, then the importance, in growth, of the regulatory hormone stands out, for the relative acceleration or retardation, as the case may be, is very marked. This too is a constitutional factor, but in a different sense, for the production of the regulator is probably local and its distribution is effected by the circulating medium, whereas the growth potential is a quality of the tissue cells in general.

Experiments to test the hypothesis regarding the source of the regulating factor have been devised and will be carried out as soon as material is available.

Summary.—The fore limb of Amblystoma punctatum develops during the embryonic period, but that of A. tigrinum relatively much later. The

latter, however, preserving its proportion to the body as a whole, attains ultimately a much greater absolute size.

The fore limb bud of A. punctatum, engrafted in normal position on a tigrinum embryo, develops at first much more rapidly than the natural limb on the opposite side of the host, but is retarded in comparison with the limb of the donnor left in place. In the reciprocal transplantation the tigrinum limb bud remains for a time small, but its growth is soon greatly accelerated and the limb later becomes much larger than that of either species in its normal surroundings.

These results may be explained by the assumption that two factors are concerned in the growth process: the growth potential, a property of the cells of the graft; and a regulator, probably an internal secretion of the host, carried to the limb through the circulation.

[1] Cope, E. D. 1889. "The Batrachia of North America." *Bull. U. S. Nat. Mus.*, No. 34.

[2] Harrison, Ross G. 1918. "Experiments on the Development of the Forelimb of Ambylstoma, a Self Differentiating Equipotential System." *J. Exp. Zoöl.*, **25**.

[3] Detwiler, S. R. 1920a. "On the Hyperplasia of Nerve Centers Resulting from Excessive Peripheral Loading." *Proc. Nat. Acad. Sci.*, **6**, pp. 96–101. Also 1920b. "Functional Regulation in Animals with Composite Spinal Cords." *Ibid.*, pp. 695–700.

SETS OF COMPLETELY INDEPENDENT POSTULATES FOR CYCLIC ORDER[1]

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The "universe of discourse" in this paper consists of all systems (K,R), where K is a class of elements, $A,B,C,\ldots$, and R is a triadic relation, denoted by R(ABC), or simply ABC.

Within this universe of discourse, two classes of systems (K,R) are of special importance; first, the *betweenness-systems* characterized by any one of twelve sets of postulates, studied elsewhere;[2] and second, the *cyclic-order-systems* characterized by any one of three sets of postulates studied in the present paper.

The three sets of postulates for cyclic order are:

(1) E, B, C, D, 2,
(2) E, B, C, D, 3,
(3) E, B, C, D, 9;

each set consisting of five postulates selected from a certain basic list (see below).

The most familiar example of a cyclic-order-system—that is, a system (K,R) in which the class K is cyclically ordered by the relation R—is the system in which K is a class of points on the circumference of a circle, and R(ABC) means that the points A, B, C, are distinct and that the arc A-B-C, read clockwise, is less than 360°.

Basic List of Postulates for Cyclic Order.—The numbering of these postulates is made to conform with the numbering already used in the papers on betweenness. Postulates E, B, C, D concern three elements; postulates 2, 3, 9 concern four elements.

POSTULATE E. *If A, B, C, are distinct, then ABC implies BCA.*

POSTULATE B. *If A,B,C, are distinct, then at least one of the orders ABC, BCA, CAB, CBA, ACB, BAC, is true.*

POSTULATE C. *If A,B,C, are distinct, then ABC and ACB cannot both be true.*

POSTULATE D. *If ABC is true, then A,B, and C are distinct.*

POSTULATE 2. *If A,B,X,Y, are distinct, and XAB and AYB, thenXAY.*

POSTULATE 3. *If A,B,X,Y are distinct, and XAB and AYB, then XYB.*

POSTULATE 9. *If A,B,C,X are distinct, and ABC is true, then at least one of the orders ABX and XBC will be true.*

This basic list is essentially exhaustive as far as "general laws" are concerned ("existence postulates" not being considered); but it contains a number of redundancies, as indicated in the following *theorems on deducibility:*

THEOREM 2_1. *Postulate 2 follows from E,C,9.*

To prove: If XAB and AYB, then XAY.

By E, AYB gives BAY. By 9, XAB with Y gives either XAY or YAB. But if YAB, then, by E, BYA, which conflicts with BAY, by C. Hence XAY.

THEOREM 2_2. *Postulate 2 follows from E, 3.*

To prove: If XAB and AYB, then XAY.

By E, XAB gives BXA; and by E, AYB gives YBA. Then by 3, YBA and BXA give YXA, whence, by E, XAY.

THEOREM 3_1. *Postulate 3 follows from E,C,9.*

To prove: If XAB and AYB, then XYB.

By E, XAB gives ABX. By 9, ABX with Y gives either ABY or YBX. But ABY conflicts with AYB, by C. Therefore YBX, whence, by E, XYB.

THEOREM 3_2. *Postulate 3 follows from E, 2.*

To prove: If XAB and AYB, then XYB.

By E, XAB gives BXA; and by E, AYB gives YBA. Then by 2. YBA and BXA give YBX, whence, by E, XYB.

THEOREM 9_1. *Postulate 9 follows from E,B,2.*

To prove: If *ABC*, then either *ABX* or *XBC*.

By E, *ABC* gives *BCA*. By B and E, either *ABX* or *XBA* must be true. But by 2, *XBA* with *BCA* leads to *XBC*. Hence either *ABX* or *XBC*.

THEOREM 9_2. *Postulate 9 follows from E,B,3.*

To prove: If *ABC*, then either *ABX* or *XBC*.

By E, *ABC* gives *CAB*. By B and E, either *ABX* or *AXB* must be true. But by 3, *CAB* with *AXB* leads to *CXB*, and hence, by E, to *XBC*. Therefore either *ABX* or *XBC*.

These theorems show that the three sets of postulates are equivalent.

Examples for Proving Independence.—We now exhibit examples of systems (K,R) having the properties described in the following tables, in which a plus sign (+) indicates that a postulate is satisfied, and a minus sign (−) that it is not satisfied.

Example 0 shows that all the postulates are *consistent*.

Examples 1, 2, 4, 16, 8 show that the five postulates of each set are independent in the ordinary sense, and examples 0–31 show that the five postulates of each set are *completely* independent in the sense of E. H. Moore.

The remaining examples enable us to see that among the seven postulates of our basic list there are no theorems on deducibility other than those proved above. Thus:

Postulate 2 is not deducible from E,B,C,D (Ex. 8); nor from E,B,D,9 (Ex. 36); nor from B,C,D,3,9 (Ex. 32).

Postulate 3 is not deducible from E,B,C,D (Ex. 8); nor from E,B,D,9 (Ex. 36); nor from B,C,D,2,9 (Ex. 33).

Postulate 9 is not deducible from E,B,C,D (Ex. 8); nor from E,C,D,2,3 (Ex. 35); nor from B,C,D,2,3 (Ex. 34).

Also, no one of the postulates E,B,C,D, is deducible from the remaining six (Exs. 1, 2, 4, 16).

Two General Theorems on Cyclic Order.—The following general theorems on cyclic order are of interest.

THEOREM I. *If A, B, C are distinct elements, in the order ABC, and if X is any other element, distinct from A, B, and C, then one and only one of the orders AXB, BXC, CXA, will be true.* (From E, C, 9).

In brief any cyclically ordered class is divided by any three of its elements into three non-overlapping subclasses.

Proof. In the first place, at least one of the orders *AXB, BXC, CXA* must be true. For, by 9, *ABC* with *X* gives either *ABX* or *XBC*, whence, by E, either *BXA* or *CXB*. But by 9, *BXA* with *C* gives *BXC* or *CXA;* and by 9, *CXB* with *A* gives *CXA* or *AXB*. Hence in any case, either *AXB* or *BXC* or *CXA* will be true.

The requisite examples are the following, the class K consisting in each case of four elements, 1, 2, 3, 4, and the relation R being defined by listing explicitly the true triads in each case.

Ex. 0. 123, 231, 312; 124, 241, 412; 134, 341, 413; 234, 342, 423.
Ex. 1. 123, 124, 134, 234.
Ex. 2. No triads true.
Ex. 3. 123, 124.
Ex. 4. All twenty-four permutations true.
Ex. 5. 123, 124, 134, 234; 132, 142, 143, 243.
Ex. 6a. 123, 231, 312; 321, 132, 213; 124, 241, 412; 421, 214, 142; 234, 342, 423; 432, 324, 243.
Ex. 6b. 123, 231, 312; 321, 132, 213.
Ex. 7. 123, 132, 124, 134.
Ex. 8. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243.
Ex. 9. 123, 214, 134, 243.
Ex. 10. 123, 231, 312; 214, 142, 421.
Ex. 11. 123, 243.
Ex. 12. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243; 321, 132, 213.
Ex. 13. 213, 231; 124, 142; 431, 413; 243, 234.
Ex. 14. 123, 231, 312; 321, 213, 132; 214, 142, 421.
Ex. 15. 123, 132; 413.
Exs. 16–31. Same as Exs. 0–15, with the triad 444 added.
Ex. 32. 123, 142, 143, 243; 423.
Ex. 33. 123, 341, 243, 124; 241.
Ex. 34. 123, 142, 234, 134.
Ex. 35. 123, 231, 312.
Ex. 36. 123, 231, 312; 321, 213, 132; 421, 214, 142; 431, 314, 143; 234, 342, 423; 432, 324, 243.

Ex.	E	B	C	D	2 3 9
0	+	+	+	+	+
1	−	+	+	+	+
2	+	−	+	+	+
3	−	−	+	+	+
4	+	+	−	+	+
5	−	+	−	+	+
6*	+	−	−	+	+
7	−	−	−	+	+
8	+	+	+	+	−
9	−	+	+	+	−
10	+	−	+	+	−
11	−	−	+	+	−
12	+	+	−	+	−
13	−	+	−	+	−
14	+	−	−	+	−
15	−	−	−	+	−
16	+	+	+	−	+
17	−	+	+	−	+
18	+	−	+	−	+
19	−	−	+	−	+
20	+	+	−	−	+
21	−	+	−	−	+
22*	+	−	−	−	+
23	−	−	−	−	+
24	+	+	+	−	−
25	−	+	+	−	−
26	+	−	+	−	−
27	−	−	+	−	−
28	+	+	−	−	−
29	−	+	−	−	−
30	+	−	−	−	−
31	−	−	−	−	−

*Examples 6a and 22a satisfy postulate 9; Examples 6b and 22b satisfy postulates 2 and 3.

Ex.	E B C	D	2 3 9
32	−++	+	−++
33	−++	+	+−+
34	−++	+	++−
35	+−+	+	++−
36	++−	+	−−+

In the second place, if any one of these three relations is true, both the other two will be false. For example, let AXB be true. Then by E, XBA and BAX. Now by 9, XBA with C gives either XBC or CBA, whence, by E, either BCX or ACB; but ACB conflicts with ABC, by C; hence BCX. Again, by 9, BAX with C gives either BAC or CAX; but BAC leads, by E, to ACB, which conflicts with ABC, by C; hence CAX. But if BCX and CAX are true, then by C, BXC and CXA must be false.

THEOREM II. *If A, B, C are distinct elements, in the order ABC; and if X, Y, Z are three other distinct elements such that AXB, BYC, CZA; then XYZ.* (From E, C, 9.)

Proof. By 9, ABC with Y gives either ABY or YBC; but YBC leads, by E, to BCY, which conflicts with BYC, by C; therefore ABY; whence by E, YAB. Then by 9, YAB with X gives either YAX or XAB; but XAB leads, by E, to ABX, which conflicts with AXB, by C; therefore YAX; whence by E, AXY.

By E, ABC gives BCA, and by 9, BCA with Y gives either BCY or YCA; but BCY conflicts with BYC, by C; therefore YCA; whence by E, CAY. Then by 9, CAY with Z gives either CAZ or ZAY; but CAZ conflicts with CZA, by C; therefore ZAY; whence by E, AYZ. Then by 9, AYZ with X gives either AYX or XYZ.

But AYX conflicts with AXY, by C. Hence XYZ.

The Relation Between Cyclic Order and Betweenness.—If we interchange two letters in Postulate E, so that it becomes

POSTULATE A. *If A, B, C are distinct, then ABC implies CBA;* then the five postulates A, B, C, D, 9 form a set of completely independent postulates for *betweenness* (*loc. cit.*, set 12).

[1] Presented to the American Mathematical Society on the following dates: the first set, December 28, 1916 [for brief abstract, see E. V. Huntington, "A Set of Independent Postulates for Cyclic Order," These PROCEEDINGS, **2**, pp. 630–631, 1916]; the second set (with proofs of complete independence for the first two sets), October 27, 1923; the third set, December 28, 1923.

[2] For the first eleven sets of postulates for betweenness, see E. V. Huntington and J. R. Kline, "Sets of Independent Postulates for Betweenness," *Trans. Amer. Math. Soc.*, **18**, pp. 301–325, 1917. For the twelfth set, involving the new postulate 9, see a forthcoming paper by E. V. Huntington, "A New Set of Postulates for Betweenness, with Proof of Complete Independence," presented to the Society, December 28, 1923, and to appear in *Trans. Amer. Math. Soc.*

THE DEVELOPMENT OF A FREQUENCY FUNCTION AND SOME COMMENTS OF CURVE FITTING

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The development of any function into a series of any type may serve either of two purposes: (1) theoretical or (2) practical. Thus the theory of power series may serve as a basis for the theory of functions of a complex variable and the theory of Fourier series is of importance for the study of certain very general types of functions of a real variable. In such cases the existence of the expansion as a convergent infinite process is the prime requisite (unless indeed we are willing to discuss the more difficult subject of divergent processes). The practical point of view is different: the matter of prime importance here is rather the rapid convergence (or pseudo-convergence) to the function throughout the domain of interest.

In statistics methods of expansion have been proposed which are based upon the Gaussian or normal frequency function.[1] A very general theorem is stated by Fisher (p. 203) to the effect that any frequency function $\varphi(z)$ which together with its first two derivatives is finite and continuous in the interval from $-\infty$ to $+\infty$ and which vanishes together with its derivatives for $= \pm \infty$ may be developed into an infinite series of the form

$$\varphi(z) = \sum c_i e^{-z^2/2} H_i(z), \qquad i = 1, 2, \ldots$$

where $H_i(z)$ is the Hermite polynomial of order i. To be useful practically the coefficients c_i must fall off rapidly in numerical magnitude. This may be expected to be the case where the function $\varphi(z)$ is approximately of the normal type $e^{-z^2/2}$; the interesting thing is to observe how seriously $\varphi(z)$ may depart from this form and still give a satisfactory rapid convergence.

Now it appears so to happen that for a good many statistical problems of economics the law of frequency is not at all close to $e^{-z^2/2\sigma^2}$ but is very close to $e^{-|z/\vartheta|}$ where the mean deviation ϑ from the median (or from the mean in symmetric distributions) replaces the standard deviation σ from the mean. The function $e^{-|z/\vartheta|}$ though continuous has a discontinuity at $x = 0$ in its first derivative and therefore fails to satisfy the conditions for expansion into an A-type series. Thus if the statistical data when drawn as a frequency curve give internal evidence of having a "beak" maximum instead of a smooth maximum there may be little consolation in the considerable amount of work necessary to compute the A-type development.

Perhaps the easiest way to examine the matter is to apply the formal method to the expansion[2] of $e^{-\sqrt{2}|z|}/\sqrt{2}$. The expansion is

$$\varphi(z) = \frac{1}{\sqrt{2}} e^{\sqrt{2}|z|} = \sum c_i \varphi_i(z) = \sum c_i \frac{1}{\sqrt{2\pi}} e^{-z^2/2} H_i(z),$$

where

$$H_i(z) = z^i - \frac{i(i-1)}{2.1!} z^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2^2.2!} z^{i-4} - \ldots$$

and

$$c_i = \frac{\sqrt{2}.\int_0^\infty e^{-\sqrt{2}z} H_i(z)dz}{(-1)^i.i!} = \frac{1}{2^k}\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots \pm \frac{1}{k!}\right]$$

where $i = 2k$ is even and $c_1 = 0$ where i is odd. This gives

$$\frac{1}{\sqrt{2}} e^{-\sqrt{2}|z|} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\left[1 + \frac{1}{8}\left(z^4 - 6z^2 + 3\right) + \right.$$

$$\left. \frac{1}{24}\left(z^6 - 15z^4 + 45z^2 - 15\right) + \frac{3}{128}\left(z^8 - 28z^6 + 210z^4 - 420z^2 + 105\right) + \ldots\right].$$

From Fisher's table (p. 280–1) the values may be computed for the first three terms (the last term goes beyond his table).

The first approximation given by the expansion is

$$\frac{1}{\sqrt{2}} e^{-\sqrt{2}|z|} = \frac{1}{\sqrt{2\pi}} e^{-\sqrt{2}|z|}.$$

As should be expected, the normal curve lies too low for small and for large values of the variable, too high for intermediate values; the curves cross at ± 0.49 and ± 2.34. The second approximation,

$$\frac{1}{\sqrt{2}} e^{-\sqrt{2}|z|} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\left[1 + \frac{1}{8}\left(z^4 - 6z^2 + 3\right)\right]$$

does a little better in spots but exhibits a secondary maximum near $z = 2^1/_2$ and is bad in other spots—perhaps on the whole it is worse. Including the next term in the expansion makes matters much worse, because values near the origin are far too small, those near $z = 1$ are too large, those near $z = 2^1/_4$ are relatively large and actually negative. Taking still more terms makes the "approximation" still worse. In a graph (fig. 1) on semi-log paper the exponential function appears as a straight line (plotting for positive z only). If the "approximations" are plotted on this paper the relative errors (not the absolute errors) are exhibited, and it is these that are of the greater importance.

That a satisfactory convergent representation of the Charlier A-type expansion should not be expected is a mathematical truism, because the conditions which legitimate the expansion are not fulfilled. It is, however,

also mathematically trite that by an arbitrarily small modification of $e^{-|z/\vartheta|}$ in the immediate neighborhood of $z = 0$ the function may be changed into one which does have continuous first and second derivatives and for which the stated conditions of the theorem would be fulfilled. Moreover such a minute modification could in no way introduce any inadequacy in the portrayal of statistical data. Now if the Charlier A-

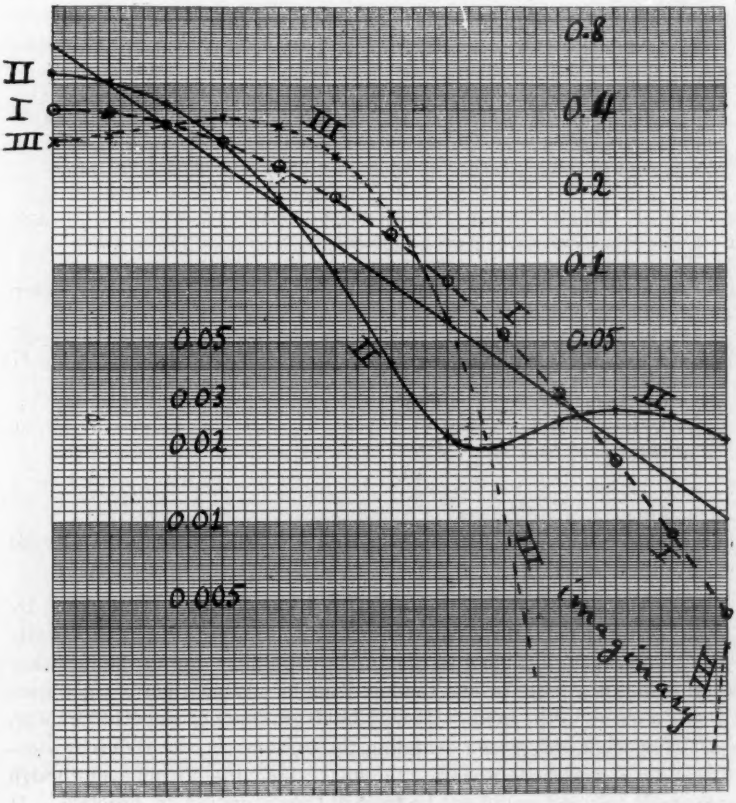

FIGURE 1

The straight line is the given frequency function plotted on an arith-log grid. Curves I, II, and III are the successive "approximations." Curve III goes to minus infinity and has an imaginary section.

type expansion of a frequency function with a "beak" maximum started to converge (like an asymptotic series) and became divergent only in its later terms, it is conceivable that the possibility of such a modification

of the "beak," which would of course leave the first terms of the expansion practically unaltered, might serve as a statistical justification for the use of the first few terms of the expansion for practical purposes in the approximate representation of the function. Unfortunately the fact is, as the illustration shows, that for such a type of frequency function the A-type expansion starts right away to diverge and hence its use is of doubtful value whenever the statistical data define a frequency function of even approximately this type.[3]

PET.	OBS.	I	II	III	IV	V	VI	P. E.
5	133	132.6	131.6	136.9	131.8	134.9	134.6	1/3
6	55	55.2	55.2	48.5	66.1	51.6	53.2	1/2
7	23	22.9	24.5	22.6	20.6	22.5	21.1	2/3
8	7	7.1	15.5	9.6	6.4	9.5	8.3	3/4
9	2	3.0	1.6	3.4	2.2	2.9	3.3	4/3
10	2	0.1	0.2	0.8	0.8	0.6	1.3	1

First column: Number of petals. Second column: Observed number of occurrences.

I. Fit by A-type expansion up to and including the term in φ_6: Fisher, p. 232.

II. Fit by A-type expansion $\varphi = 264.2\varphi_0 - 89.9\varphi_3 - 5.2\varphi_4$; Fisher, p. 232.

III. Fit by Pearson to the appropriate one of his types of frequency curves; Fisher, p. 232.

IV. Fit by logarithmic transformation $F = \frac{183.2}{0.5445\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\log(x-4)-0.044}{0.5445}\right]^2}$; Fisher, p. 260.

V. Fit by B-type curve: $F = 222\psi(x-5) + 31.5\,\Delta^2\psi(x-5); \psi(x) = e^{-0.631}(0.631)^x : x!$; Fisher, p. 274.

VI. Fit by $\text{Log}\,\frac{F}{10} = -0.4027\,(x-5) + 2.129$ or $F = 134.6 \times 10^{-0.4027(x-5)} = 134.6(.3956)^{(x-5)}$.

Last column: The fraction of the probable error of sampling by which the values VI depart from those observed.

Usually the frequency function $\varphi(z)$ is not known but is represented by a table of statistical data giving the integrated values of $\varphi(z)$ in a series of class intervals and subject to inevitable variability due to the fact that the table is but a more or less random sample. What then is the object of fitting a curve to the table? It may be to graduate or smooth the table, to get somewhat rid of the sampling fluctuations in the different class intervals. It may be to illustrate the method of curve fitting even though the example selected would not be treated thus if arising in practice. It may be to obtain analytic expressions or graphical forms which may be employed for the hazardous purpose of extrapolation which means, in the case of a time-series, forecasting the future. But I conceive that the true object of fitting an analytic formula to a set of data is to discover a law of nature.

For the revelation of natural law series expansions are of minor utility; rather are they for correcting or amending a law true in the main. Much

the same applies to the relatively complicated curve types suggested by Pearson. A simple rough fit may be suggestive of a working hypothesis which may lead to the discovery of a natural law where a complicated close fit may be its detail obscure; Newton had Kepler's three laws to aid him in his discovery of the law of gravitation, and may have been fortunate in not having too much more. I may illustrate by making a simple and comparatively laborless fit in the famous case of the number of petals in Ranunculus. The table summarizes the values.

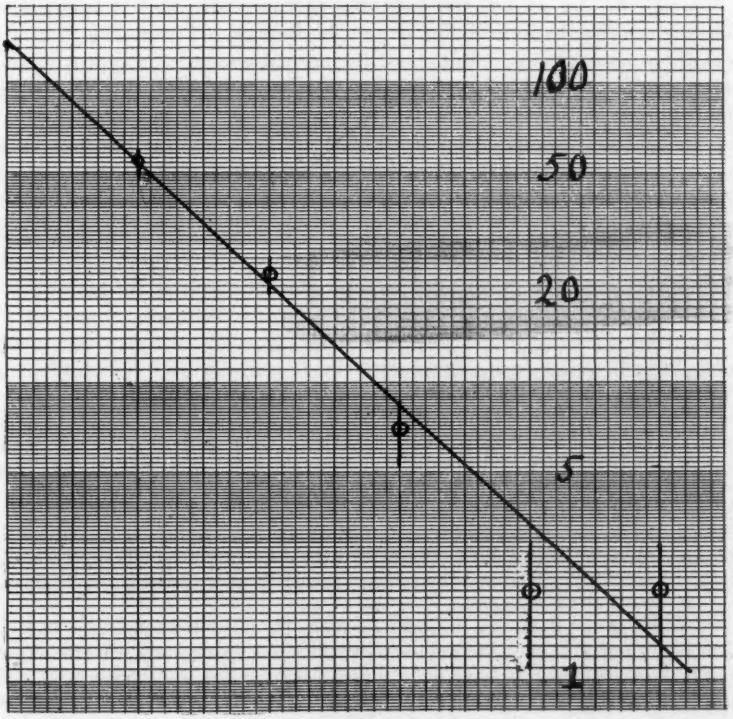

FIGURE 2

The points surrounded by circles are the observations plotted on the arith-log grid. The vertical lines indicate the arrangement of chance fluctuations. The straight line represents a graphical fit to the observed points.

For details of the fits I–V reference must be made to the citations. In I the expansion was carried to φ_6 which gave 5 constants c_0, c_2, c_4, c_5, c_6 available to fit 6 points after a quasi-fit had been made by so selecting the origin and the scale that $c_1 = c_2 = 0$; in II there are 3 constants, c_0, c_3, c_4 disposable.[4] The method of determining my own fit VI was to plot

(fig. 2) the observations on arith-log paper, marking vertically about each the latitude allowed by the probable error of sampling, drawing a straight line nearly through the points having regard to the latitude indicated, and fitting an equation to the line. Whether this simple fit is as good as the others I will not say, but it is good enough because the observed values depart from the curve by less than the probable error of sampling in 4 cases out of 6, and the two departures that are greater occur in numbers so small that they are really meaningless.

Finally my fit VI has the desirable characteristic that it suggests at once a working hypothesis for it may be interpreted as stating that the chance of an additional petal above the normal number 5 is approximately 4/10. It may not be a good biological working hypothesis that the chance of a supermumerary petal is a definite fraction with the law of the product of independent probabilities applicable to determining the chances of several supernumeraries. Perhaps a complicated and uninterpretable formula is better in that it does not suggest working hypotheses looking towards the discovery of natural laws; but it is difficult for me to dismiss the impression that the discovery of laws in economics and biology will follow the same general lines as in those sciences already further reduced to law.

[1] For Charlier's A-type see Arne Fisher, *Mathematical Theory of Probabilities*, Macmillian, 1922, especially Chaps. XIII–XVII.

[2] For this particular form (dependent merely on choice of units) $\sigma = 1$.

[3] The point might appear too obvious; but statisticians are trying the expansion in such cases without apparently realizing its danger. See E. B. Wilson, *J. Amer. Statist. Ass.*, **18**, 1923 (841–851) and W. L. Crum, *Ibid.* (607–614). Arne Fisher, *Ibid.* (794) makes the suggestion that the logarithmic transformation of the variate would help, and he has worked so much with the method that his judgment ought to be trusted; but I cannot see how any transformation which is continuous with continuous derivatives can resolve a "beak" in such a way as to legitimize the use of A-type expansion; the logical difficulty would therefore seem to remain even though Fisher's suggestion should ameliorate the practical inconveniences.

[4] I really do not know how to count constants for the A-type expansion because although c_1 and c_2 are not undetermined constants in the least-squares sense the adjustments of origin so that $c_1 = 0$ and the selection of scale so that $c_2 = 0$ do seem to imply a certain fitting of these to the problem in hand. Moreover I will not pause to indicate the peculiar vagaries of the curves I and II between the integral number of petals, except to observe that if this were a problem involving a continuous instead of a discrete variate either the fit would be bad or the table gives no indication of the intermediate frequencies.